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Commuting-flow symmetries and common solutions to differential equations with common symmetries

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Abstract. We point out that in certain cases, all the differential equations (for given indipendent and dependent variables) possessing a given symmetry necessarily share a common solution. Under weaker conditions, all such differential equations have a solution—in general, different for different equations—characterized by a common symmetry. We characterize this situation and the common solution, or the common symmetry of solutions, and give concrete examples.

0. Introduction

We are all familiar with elementary occurences of a remarkable phenomenon: in some cases, *all* the differential equations possessing a given symmetry share a common solution.

As trivial examples of this fact it suffices to think of reflection-invariant ordinary differential equations (ODEs) in $\mathcal{R}^1$, $\dot{x} = f(x,t)$ (these are identified by $f(x,t) = -f(-x,t)$), which all share the common solution $x(t) \equiv 0$; or also reflection and/or rotation invariant ODEs in $\mathcal{R}^n$, which again share the common (trivial) solution $x(t) \equiv 0$.

The purpose of this short note is to point out that this phenomenon can be present in far less trivial cases, for both ODEs and partial differential equations (PDEs); and to give a way to characterize the common solutions, at least for a relevant class of symmetries, i.e. the 'commuting-flow symmetries' defined below (which are natural in a dynamical systems geometric approach).

This question can be tackled making use of Michel's theory (see below), and indeed our result can be seen as a simple corollary—maybe reaching unexpected fields—of the original Michel theorem.

A classical result of Michel [1] ensures that, given a finite-dimensional smooth manifold M with an action of a compact Lie group G, we can identify the points $x \in M$ which are critical for *all* the G-invariant potentials $V(x)$. The investigation of Michel was motivated principally by fundamental problems in physics (the theory of strong interactions) [2], but the theorem in itself is of a geometric nature and of a much wider applicability to a range of problems involving *spontaneous symmetry breaking*.

The Michel's result can be suitably generalized to variational problems in function spaces [3], and it has been recently extended to consider gauge invariant functionals [4].

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On the other side, the proof given by Michel in [1] for his theorem immediately generalizes to consider equivariant dynamics defined by a vector field on M and its fixed points; it also guarantees, more generally, the invariance under the equivariant dynamics of a family of subspaces of M (identified by sharing a common isotropy subgroup for the G-action). In this way, one could also readily recover a number of fundamental results in equivariant dynamics and equivariant bifurcation theory, such as the reduction lemma of Golubitsky and Stewart [5] and the equivariant branching lemma of Cicogna and Vanderbauwhede [6], originally obtained with no use of Michel's theory [7].

It is thus quite natural to ask if, beside the extension of the variational aspects of Michel theory to infinite-dimensional spaces [4], one could not obtain also a similar extension for its equivariant dynamics aspects. The purpose of this note is to show that this is indeed the case, and namely that we can identify subsets of a given function space $\mathcal{F}$ which are invariant under any dynamics on $\mathcal{F}$ which satisfy certain symmetry requirements with respect to an algebra $\mathcal{G}$ of vector fields; moreover, proceeding pretty much as in the finite-dimensional case, we can identify functions $f_0 \in \mathcal{F}$ which are solutions for *any* differential equation Δ which is $\mathcal{G}$-invariant (in a sense to be precisely defined in the following).

Our discussion makes use of ideas and formalisms from the symmetry theory of differential equations [8–11], which we assume is known—at least in its basics—to the reader. Also, our discussion will be conducted at a rather formal (in the sense of non-rigorous) level; thus, in particular, we will assume the existence of flows generated by the (generalized) vector fields to be considered in the following.

1. Evolutionary vector fields and symmetry

Let us consider a smooth manifold $M = \mathcal{X} \times \mathcal{Y}$, where the $x \in \mathcal{X} \subseteq \mathcal{R}^q$ and the $u \in \mathcal{Y} \subseteq \mathcal{R}^p$ will be thought as, respectively, the independent and dependent variables for the differential equations to be considered in a moment.

We consider a vector field Y on M, which we write as

$$Y = \sum_{i=1}^{q} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{p} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{1}$$

(with ξ, φ smooth functions on M); as is well known [8], to any such vector field we can associate an *evolutionary representative*; this will be denoted by X and is given by

$$X = \left[\varphi^\alpha(x,u) - \sum_i \xi^i(x,u)u_i^\alpha\right]\partial_\alpha \equiv Q^\alpha[u]\partial_\alpha. \tag{2}$$

Here we write ∂_α for $\partial/\partial u^\alpha$, and u_i^α for $\partial u^\alpha/\partial x^i$ (these notations will be understood throughout in the following; from now on we will also understand summation over repeated indices). The notation $Q[u]$—used extensively in the following—means that Q is a function of x, u, and u derivatives of any order (although in this case these are only of first order). We will also write X as X_Q, to stress the association with the vector $Q[u]$; more generally, X_P will denote the evolutionary vector field $X_P = P^\alpha[u]\partial_\alpha$.

It should be recalled that, while Y is a legitimate vector field on M, its evolutionary representative X should be considered with some extra care; we can either consider it as a notation for a vector field in the function space $\mathcal{M}$ (the space of smooth functions from $\mathcal{X} \subseteq \mathcal{R}^q$ to $\mathcal{Y} \subseteq \mathcal{R}^p$), such that $(I + \varepsilon X_Q)$ transforms $f(x)$ into the function

$$\tilde{f}^\alpha(x) = f^\alpha(x) + \varepsilon[\varphi^\alpha(x, f(x)) - \xi^i(x, f(x))f_i^\alpha(x)] \tag{3}$$

either as a generalized vector field acting in the jet space associated with M; in this case, one should more precisely consider the infinite-order prolongation X^* of X, acting in the infinite-order jet space $J^*(M)$, in order to have a proper vector field.

It should also be stressed that, for the argument to be considered below, one could start directly from X^*, i.e. consider generalized vector fields rather than be limited to consider only representatives of geometric vector fields.

Let us now consider a system of p differential equations for functions $u^\alpha = u^\alpha(x) \in \mathcal{M}$; this will be written as $\Delta[u] = 0$ (this will be a vector Δ^α, with $\alpha = 1, \ldots, p$; we will say for simplicity 'the equation' Δ to refer to this system). With the equation Δ we associate an evolutionary generalized vector field X_Δ given by

$$X_\Delta = \Delta^\alpha[u]\partial_\alpha \tag{4}$$

as it was in the case for X_Q, this can be seen either as a vector field on $\mathcal{M}$, either as denoting the infinite prolongation X_Δ^* acting in $J^*(M)$.

We recall [8] that X_Q is a *symmetry* of Δ if

$$X_Q^*(\Delta) = 0 \tag{5}$$

is satisfied whenever $\Delta = 0$, and a *strong symmetry* of Δ if (5) is identically satisfied.

2. Commuting flow symmetries

Given two evolutionary generalized vector fields (EGVF), $X_P = P^\alpha[u]\partial_\alpha$ and $X_Q = Q^\alpha[u]\partial_\alpha$, we can consider their commutator $[X_P, X_Q]$; this will be again an EGVF $Y_R = R^\alpha[u]\partial_\alpha$, with

$$R^\alpha[u] = X_P^*(Q^\alpha) - X_Q^*(P^\alpha). \tag{6}$$

It is easy to check [8] that this gives indeed a proper commutator, i.e. the usual properties of the commutator are satisfied.

Given a differential equation Δ, and an EGVF X_Q, we say that X_Q is a *commuting flow (CF) symmetry* for Δ if the associated EGVF commute (in the above sense), i.e. if

$$[X_Q, X_\Delta] = 0. \tag{7}$$

For a given equation Δ, we denote by $\mathcal{G}_\Delta$ the set of EGVF which commute with X_Δ (this is easily seen to be a Lie algebra, with X_Δ belonging to its centre). Conversely, for a given EGVF X_Q we denote by $\mathcal{I}_Q$ the set of differential equations which have X_Q as a CF symmetry; again, the X_Δ corresponding to $\Delta \in \mathcal{I}_Q$ form a Lie algebra.

Lemma 1. If X_Q is a CF symmetry for Δ, then it is also a symmetry of Δ in the ordinary sense.

Proof. Indeed, on $\Delta = 0$ we have that both X_Δ and X_Δ^* vanish, as it is clear since $X_\Delta = \Delta^\alpha[u]\partial_\alpha$ and $X_\Delta^* = \sum_J (D_J \Delta^\alpha[u])(\partial/\partial u_J^\alpha)$ (here J is again a multi-index). Thus, $X_\Delta^*(Q) = 0$ on $\Delta = 0$ for any Q, and the vanishing of $\left[X_Q, X_\Delta\right]$ is equivalent to the vanishing of $X_Q^*(\Delta)$, which is the condition (4) for X_Q to be a symmetry of Δ in the ordinary sense. $\square$

3. Invariance of fixed-space and common solutions to symmetric equations

With any $X_Q : \mathcal{M} \to \mathrm{T}\mathcal{M}$ we associate the set $F(Q) \subseteq \mathcal{M}$ of functions which are fixed points for the vector field X_Q; equivalently, this means that $Q[f] = 0$ for $f \in F(Q)$.

Lemma 2. If X_Q is a CF symmetry of Δ, then X_Δ leaves $F(Q)$ globally invariant, i.e. $X_\Delta : F(Q) \to \mathrm{T}F(Q)$.

It may be appropriate—in order to avoid any confusion—to point out that, here and in the following (see e.g. lemma 3 and the proposition below), when we say that $F(Q)$ is 'globally invariant' under a vector field X we mean that the finite action of X, e^{sX}, transforms $F(Q)$ into itself for any value of the parameter s; this is also equivalent, to saying that X is a tangent vector field on $F(Q)$, i.e. $X : F(Q) \to \mathrm{T}F(Q)$ as mentioned in the lemma.

Proof of lemma 2. Let us consider, for the sake of clarity, the integrated version of the commutation relations (assuming suitable existence and unicity conditions for the flows of the vector fields); we have—with λ, μ real parameters—that

$$\mathrm{e}^{\lambda Q[u]}\mathrm{e}^{\mu\Delta[u]}(f) = \mathrm{e}^{\mu\Delta[u]}\mathrm{e}^{\lambda Q[u]}(f). \tag{8}$$

From $f \in F(Q)$ it follows that $\mathrm{e}^{\lambda Q}(f) = f$, and thus we have that

$$\mathrm{e}^{\mu\Delta[u]}(f) \in F(Q) \qquad \forall\mu \tag{9}$$

from which the lemma follows immediately. □

Corollary. If there exists an $f_0 \in F(Q)$ isolated in $F(Q)$, then necessarily f_0 is solution to any $\Delta \in \mathcal{I}_Q$.

Proof. In this case, f_0 is a fixed point of $\Delta[u]$; but this is equivalent to being a solution to Δ. □

Note that when we speak of an 'isolated' function, we are implicitely assuming $\mathcal{M}$ is equipped with some natural topology; in the examples below, in which we consider a space of L^2 functions, this will be the L^2 norm. Actually, we could consider any topology in which the flows of X_Q and X_Δ are continuous.

The above lemma and corollary are immediately generalized to the case of a whole algebra of CF symmetries; we denote by $\mathcal{G} \equiv \mathcal{G}_\Delta$ this (possibly, but not necessarily, Abelian) algebra, and say that $\mathcal{G}$ is a CF symmetry algebra for Δ if $[X_Q, X_\Delta] = 0$ for any $X_Q \in \mathcal{G}$. In this case we denote by $F(\mathcal{G})$ the set of functions invariant under all the $X_Q \in \mathcal{G}$, and by $\mathcal{I}_\mathcal{G}$ the set of equations for which $\mathcal{G}$ is a CF symmetry algebra. We have then immediately the following lemma.

Lemma 3. If $\mathcal{G}$ is a CF symmetry algebra of Δ, then X_Δ leaves $F(\mathcal{G})$ globally invariant, i.e. $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$.

Corollary. If there exists a $f_0 \in F(\mathcal{G})$ isolated in $F(\mathcal{G})$, then necessarily f_0 is solution of any $\Delta \in \mathcal{I}_\mathcal{G}$.

It should be noted that it is not easy to have such an isolated $f_0 \in F(Q) \subseteq \mathcal{M}$; e.g. in the case where $Q[u]$ is a homogeneous linear function of u, we have that necessarily $F(Q)$

is a linear subspace of $\mathcal{M}$, and we cannot have isolated points (unless $F[Q]$ reduces to the zero function alone).

However, we have so far considered the whole function space $\mathcal{M}$; when we deal with differential equations, we usually have to look for solutions in some given function space, which we will denote here by $\mathcal{F}$, less general than $\mathcal{M}$; restricting to a smaller function space makes it easier to find isolated fixed points f_0.

Let us consider a closed subspace $\mathcal{F} \subseteq \mathcal{M}$, which we think as fixed once and for all. We have now to consider only those Δ such that X_Δ are vector fields on $\mathcal{F}$. We can then obtain again the equivalent of lemma 3 and its corollary given above, provided we consider now the intersection of $F(\mathcal{G}) \subseteq \mathcal{M}$ with $\mathcal{F}$.

We summarize our discussion in the following form.

Proposition. Let $\mathcal{G}$ be an algebra of EGVF on $\mathcal{M}$, and $F(\mathcal{G})$ the space of fixed points for $\mathcal{G}$; let $\mathcal{F} \subseteq \mathcal{M}$ be a subspace of $\mathcal{M}$, such that there exists a f_0 isolated in $F(\mathcal{G}) \cap \mathcal{F}$. Then, any differential equation $\Delta \in \mathcal{I}_\mathcal{G}$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits $f_0(x) \in \mathcal{F}$ as a solution.

4. Discussion and generalization

The result obtained in the previous section is clearly potentially quite relevant, but not generally applicable; the reason for this limitation lies in the fact that the existence of a nontrivial f_0 isolated in $F(\mathcal{G}) \cap \mathcal{F}$ can amount to a quite strong—and thus restrictive—condition.

Indeed, let us consider the case where the $Q[u]$ are *linear* in the u for all the $X_Q \in \mathcal{G}$; this includes in particular—but not only—the case of linear $\mathcal{G}$ action. Now $F(\mathcal{G})$ is necessarily a linear space V and therefore, with the exception of the trivial case $F(\mathcal{G}) \equiv \{0\}$, there can be no isolated $f_0 \in F(\mathcal{G})$. In this case, as already remarked above, the only way to have an isolated nontrivial $f_0 \in F(\mathcal{G}) \cap \mathcal{F}$, with $f_0 \in V$, is when $\mathcal{F}$ fixes the norm of functions, e.g. $||f|| = 1$; this is indeed how the examples in section 5 below, in which we get isolated points, fit in our frame.

We stress that the condition $\|f\| = c$ should not be seen as an artificial one, and actually it can be a physically natural one: this is e.g. the case when $f(x,t)$ represents a probability distribution, or more generally the density of some conserved quantity, and the norm $\|f\|$ is chosen to be the spatial integral of such a density, $\|f\| = \int_D f(x,t)\,\mathrm{d}x$; thus, even in the case in which we can have an isolated f_0 only through $\mathcal{F} \subseteq \{f : \|f\| = c\}$, our result has a physical interest.

After clarifying this point, we would like however to mention—and discuss informally—how our approach can give some interesting results also when we have an isolated k-dimensional (k finite) submanifold $E^k \equiv \mathcal{E} \subseteq F(\mathcal{G})$, or when $F(\mathcal{G})$ is itself a finite-dimensional (linear) space. (Note that in this paper E^k will denote a smooth manifold of dimension k, not necessarily a linear space.)

The main idea will be that, on the basis of the same kind of argument used above, we can guarantee the invariance of $\mathcal{E}$ under the dynamics: then we can consider the restriction of the flow of X_Δ to $\mathcal{E}$, which we call $X_\Delta^\mathcal{E}$, and in this way we are reduced to considering a finite-dimensional problem. We can then use all the arsenal of standard techniques and results for the finite-dimensional setting and in particular the same results that we are generalizing to the functional setting, i.e.—as mentioned in the introduction—the reduction lemma [5] and the equivariant branching lemma [6]. In this way (essentially, via the interpretation of lemma 3 as a reduction lemma in infinite dimensions) we can generalize our result for isolated points $f_0 \in F(\mathcal{G}) \cap \mathcal{F}$ to more complex isolated finite-dimensional sets $\mathcal{E} \cap \mathcal{F} \in F(\mathcal{G}) \cap \mathcal{F}$. One can thus obtain a wealth of possible specific results by considering suitable special settings, essentially reproducing the finite-dimensional theory in the framework of a functional setting.

We will now briefly mention the extensions of our results that can be obtained along these lines, at least in the simplest cases; the proofs use different versions of fixed-point theorems and will be just sketched.

For this extension, we require that there exists a compact set $D \subset \mathcal{E}$, itself invariant under the dynamics: then we are legitimate in considering the restriction of our problem to D, and can use, thanks to compactness, methods of topological nature to guarantee the existence of fixed points. Typically, such a compact invariant set D will arise as the intersection with $\mathcal{E}$ of a set which is known to be invariant due to the nature of the problem, e.g. on the basis of energy considerations: if the energy of f is a convex function of $\|f\|$ (at least for $\|f\|$ large enough), we know that the set of functions f with energy $E_f \leqslant E_0$, which is invariant under a conservative or dissipative dynamics, will be compact. It should be clear that the fixed points $x_\Delta \in D \subset E^k \equiv \mathcal{E}$ whose existence can be guaranteed in this way will be different for different equations Δ sharing the same CF symmetry, while the manifold $\mathcal{E}$—which is identified by the symmetry alone—is common to all such equations defined in the same function space.

We will denote by $\mathcal{C}_\mathcal{G}^\mathcal{E} \subset \mathcal{I}_\mathcal{G}$ the set of equations Δ for which there is a compact and connected set $D \subset \mathcal{E}$ globally invariant under X_Δ.

Lemma 4. Let $\mathcal{G}$ be an algebra of EGVF on $\mathcal{M}$, and $F(\mathcal{G})$ the space of fixed points for $\mathcal{G}$; let $\mathcal{F} \subseteq \mathcal{M}$ be a subspace of $\mathcal{M}$, such that there exists a curve $\mathcal{E} \equiv E^1 = \{f_\lambda\}$ such that $\mathcal{E} \cap \mathcal{F}$ is isolated in $F(\mathcal{G}) \cap \mathcal{F}$. Then, any differential equation $\Delta \in \mathcal{C}_\mathcal{G}^\mathcal{E}$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits a solution $f_{\lambda(\Delta)} \in (\mathcal{E} \cap \mathcal{F})$. If $\mathcal{E}$ is a line $\Lambda = \{\lambda f_0\} = R^1$, then any differential equation $\Delta \in \mathcal{C}_\mathcal{G}^\Lambda$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits a solution of the form $f_\Delta(x) = \lambda_\Delta f_0(x)$.

Proof. We give the proof in the case of a line, for ease of notation; the general one is analogous. Consider X_Δ^Λ; parametrizing $\Lambda = E^1 = \{\lambda f_0\}$ by $\lambda \in R$, we have $X_\Delta^\Lambda = \alpha(\lambda)\partial_\lambda$. The compact set $D \in \Lambda$ required by $X_\Delta \in \mathcal{C}_\mathcal{G}^\Lambda$ will be an interval $D = [\lambda_a, \lambda_b]$ (where $\lambda_a < \lambda_b$), and moreover we have $\alpha(\lambda_a) \geqslant 0$, $\alpha(\lambda_b) \leqslant 0$. We conclude that there exists a $\lambda_\Delta \in D$ such that $\alpha(\lambda_\Delta) = 0$, so that it represents a fixed point of X_Δ^Λ and hence a solution to Δ. □

It should be mentioned that we would have the same property for equations such that $X_\Delta : \mathcal{E} \to \mathrm{T}\mathcal{E}$ (provided of course $X_\Delta \in \mathcal{C}_\mathcal{G}^\mathcal{E}$) even if it is not true that $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$. In this case $\mathcal{G}$ would correspond to a kind of 'conditional CF symmetry', i.e. a symmetry valid only in some subset (in this case $\mathcal{E}$) of the function space in which we operate; in the application of Lie groups theory to differential equations [8–11], conditional symmetries [12, 13] have proved to be a specially valuable tool in obtaining concrete solutions.

Lemma 5. Let $\mathcal{G}$ be an algebra of EGVF on $\mathcal{M}$, and $F(\mathcal{G})$ the space of fixed points for $\mathcal{G}$; let $\mathcal{F} \subseteq \mathcal{M}$ be a subspace of $\mathcal{M}$, such that there exists a k-dimensional submanifold $\mathcal{E} \equiv E^k$ with $\mathcal{E} \cap \mathcal{F}$ isolated in $F(\mathcal{G}) \cap \mathcal{F}$. Then, any differential equation $\Delta \in \mathcal{C}_\mathcal{G}^\mathcal{E}$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits a solution $f_\Delta(x) \in (\mathcal{E} \cap \mathcal{F})$.

Proof. This amounts to the Brouwer fixed-point theorem for maps $h : D^k \to D^k$, see e.g. [14]. □

In many cases of interest, the equations will be known to have a 'trivial' solution, e.g. $f(x,t) \equiv 0$ (or any other known solution f_0); if this point lies in $\mathcal{E}$, all statements about the existence of a critical point in $\mathcal{E}$ are potentially trivial; moreover, when we deal with linear $Q[u]$, if we have more than one globally invariant finite-dimensional linear space, these will intersect in $\{0\}$ and thus cannot be isolated. If $f = 0$ is an unstable critical point for $X^{\mathcal{E}}_{\Delta}$, however, these problems can easily be cured by considering $\mathcal{E}\backslash\{0\}$, and correspondingly, if $\{0\} \in D$, a compact invariant submanifold $D\backslash\{0\}$, see below.

Lemma 6. Let $\mathcal{G}$ be an algebra of EGVF on $\mathcal{M}$, and $F(\mathcal{G})$ the space of fixed points for $\mathcal{G}$; let $\mathcal{F} \subseteq \mathcal{M}$ be a subspace of $\mathcal{M}$, such that there exists a k-dimensional $\mathcal{E} \equiv E^{2j+1}$ with k odd and $\mathcal{E} \cap \mathcal{F}$ isolated in $F(\mathcal{G}) \cap \mathcal{F}$. Denote by f_0 a reference solution lying in $\mathcal{E}$, and denote by $\mathcal{D}_0$ the set of equations Δ for which $X_\Delta(f_0) = 0$ and f_0 is an unstable fixed point for $X^{\mathcal{E}}_{\Delta}$. Then, any differential equation $\Delta \in \mathcal{C}^{\mathcal{E}}_{\mathcal{G}} \cap \mathcal{D}_0$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits a solution $f_\Delta(x) \neq f_0$ with $f_\Delta \in \mathcal{E}$.

Proof. This amounts again to classical results on fixed points; indeed, by an elementary change of origin in $\mathcal{E}$, we choose $f_0 = 0$; now we consider a disk $D^k \subset \mathcal{E}$ (with $k = 2j+1$) such that D^k is globally invariant under $X^{\mathcal{E}}_{\Delta}$, and the punctured disk $D^k_0 \equiv D^k\backslash\{f_0\}$. Now D^k_0 is homotopic to S^{k-1}, and by a standard topological argument [14] we can reduce it to consider vector fields on S^{k-1}; it is well known that any vector field on S^p has a fixed point if p is even, and thus the result follows. $\square$

Similar to what happens for the reduction lemma in finite dimensions [5], one could consider equations Δ_λ depending on real parameters $\lambda \in \Lambda \subseteq \mathcal{R}^n$, in such a way that the relevant CF symmetries are such for all values of $\lambda \in \Lambda$; in this case, one would extend to the present setting the results known to hold in the finite-dimensional case; once again, this just amounts to the fact that we can reduce the problem to the relevant (and finite-dimensional) fixed space $\mathcal{E}$, on which the standard analysis [5, 6] applies.

Finally, we would like to mention that in the standard application of group-theoretical methods to differential equations [8–11], one of the simplest procedures to find specific solutions to a given differential equation Δ consists of determining its symmetry algebra $\mathcal{G}_\Delta$ and then looking for solutions which are invariant under any subalgebra $\mathcal{G}_0 \subseteq \mathcal{G}$; the symmetry invariance is equivalent to an *ansatz* on the form of the solution (this can be a function of the invariants of $\mathcal{G}_0$ alone), and thus leads to a reduction of the differential equation, the more considerable the bigger the subalgebra $\mathcal{G}_0$ is. It should be noted, however, that for a general subalgebra $\mathcal{G}_0$ we are not guaranteed of the existence of nontrivial $\mathcal{G}_0$-invariant solutions. In this frame, our results can be used as a tool to be sure *a priori*—i.e. before embarking on considerably complicated computations—that for a given class of symmetries we have indeed an invariant solution.

5. Examples

Example 1. As a first example, we will deal with the fundamental solution of the heat equation. Let us consider $\mathcal{X} = R \times R_+$ and $\mathcal{Y} = R^1$; then $\mathcal{M}$ is the space of $\mathcal{C}^\infty$ functions $f : R^2 \to R$, and in this we select the space $\mathcal{F}$ of functions which satisfy, with (x,t) coordinates in $\mathcal{X}$,

$$\int_{-\infty}^{+\infty} f(x,t)\,\mathrm{d}x = 1 \tag{10}$$

and such that $f_x(x,t)$ go to zero when x go to infinity.

We consider $Y = 2t\partial_x - xu\partial_u$, and correspondingly

$$X = -(xu + 2tu_x)\partial_u. \tag{11}$$

Note that Y is one of the Lie-point symmetries of the heat equation [8–11], and it can be checked that X is a CF symmetry of it. Moreover, for Δ the heat equation, we also obviously have $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$.

It can be checked easily that the functions in $F(Q)$ are those of the form

$$f(x,t) = k(t)\mathrm{e}^{-x^2/4t} \tag{12}$$

and the intersection of this $F(Q)$ with $\mathcal{F}$ yields an isolated point, which is precisely

$$f_0(x,t) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-x^2/4t} \tag{13}$$

i.e. the fundamental solution of the heat equation.

Example 2. Let us consider $\mathcal{X} = S^1$, $\mathcal{Y} = R$, so that $\mathcal{M}$ is the space of $\mathcal{C}^\infty$ function $f : S^1 \to R$; in this we select the space $\mathcal{F}$ of functions which satisfy

$$\|f\|^2 \equiv \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x = 1. \tag{14}$$

We consider $Y = -\partial/\partial x$, and correspondingly

$$X_Q = u_x\partial_u \qquad Q[u] = u_x. \tag{15}$$

Let us now look for $F(Q)\cap\mathcal{F}$; $F(Q)$ is given by constant functions $f(x) = c$, and the constraint (14) selects the two functions $f_\pm(x) = \pm 1$. These are obviously isolated in $F(Q)\cap\mathcal{F}$.

Let us now determine the $\Delta \in \mathcal{I}_Q$; the condition $\left[X_Q, X_\Delta\right] = 0$ amounts to $\partial\Delta/\partial x = 0$. Indeed, $X_Q^* = \sum_j (D_j u_x)\partial_{u_j}$, while the first prolongation of X_Δ is given by $X_\Delta^{(1)} = \Delta\partial_u + (D_x\Delta)\partial_{u_x}$. Thus,

$$X_Q^*(\Delta) - X_\Delta^*(Q) = \sum_j (D_j u_x)\frac{\partial\Delta}{\partial u_j} - D_x\Delta. \tag{16}$$

Expanding $D_x\Delta = \Delta_x + \sum_j(\partial\Delta/\partial u_j)u_{x,j}$ (where $u_{x,j} = D_j u_x$), we get that (16) reduces to Δ_x.

Thus, $\Delta \in \mathcal{I}_Q$ are written as

$$\Delta = \Delta(u, u_x, u_{xx}, \ldots) = 0. \tag{17}$$

Now we have to check that all such Δ which, moreover, have X_Δ leaving $\mathcal{F}$ invariant admit $f_\pm(x)$ as solutions.

The requirement $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ amounts to asking that

$$\|\mathrm{e}^{\lambda X_\Delta}(f)\| = 1 \tag{18}$$

whenever $\|f\| = 1$. In the limit $\lambda \to 0^+$,

$$\mathrm{e}^{\lambda X_\Delta}(f) \simeq (I + \lambda X_\Delta)(f) \simeq f + \lambda\Delta[f] \tag{19}$$

and thus

$$\|\mathrm{e}^{\lambda X_\Delta}(f)\|^2 = \|f\|^2 + 2\lambda\langle f, \Delta[f]\rangle + \mathrm{o}(\lambda). \tag{20}$$

As (19) must hold for any $f \in \mathcal{F}$, it must in particular hold on $f_\pm$; evaluating (20) in $f_\pm$ we obtain then $\Delta(\pm 1, 0, 0, 0, \ldots) = 0$, i.e. that $f_\pm$ is a solution to Δ.

In order to mention concrete examples of problems to which the present example applies, consider stationary (i.e. equilibrium) solutions for nonlinear diffusion equations in S^1,

$$u_t = \triangle u + W(u) \tag{17'}$$

with $W(u)$ a nonlinear term, or directly for a nonlinear wave equation $\triangle u + W(u) = 0$ for an incompressible fluid in the unit interval with periodic boundary conditions.

In the first case $u(x)$ represents a probability density, so that the condition (14) must indeed be required; in the second case, considering this e.g. for water waves, $u(x)$ represents the level of the liquid at point x, and the integral (14) just represents the (normalized) total mass of the liquid; more generally, we could have any conserved field obeying the wave equation as an example of the natural occurrence of the condition (14).

Example 3. Let us consider the same setting as in the previous example, but now with $\mathcal{M} \equiv \mathcal{F}$, or $\mathcal{M} = L^2(S^1, R)$: i.e. in this case we are removing the constraint on the norm of f.

In this case, $F(Q)$ is still given by constant functions, and thus is $F(Q) = \Lambda \approx R^1$. The condition for the existence of a globally invariant disk (i.e. interval) $D^1 \subset F(Q) = R^1$ can be expressed as follows when we consider the projection X_Δ^Λ of X_Δ on $F(Q)$: this restriction can be written—using the parametrization of Λ in terms of $\lambda \in R^1$—as $X_\Delta^\Lambda = \alpha(\lambda)\partial_\lambda$; we require then that there are constant functions $f_1 = \lambda_1$ and $f_2 = \lambda_2$ (with $\lambda_1 < \lambda_2$ and λ_1, λ_2 finite) such that

$$\alpha(\lambda_1) > 0 \qquad \alpha(\lambda_2) < 0 \tag{21}$$

and under this condition the equation Δ will have a constant solution of finite norm.

If we consider equations of the form (17′), it is clear that constant solutions correspond to zeros of $W(u)$; moreover, X_Δ^Λ is just $W(u)\partial_u \equiv W(c)\partial_c$ and the above condition (21) does indeed imply the existence of a stable zero of X_Δ^Λ in the interval (λ_1, λ_2), i.e. of a stable stationary solution, constant in x, of (17′).

Example 4. We consider now an example with $u = u(x_1, x_2)$ and an algebra of CF symmetries spanned by two vector fields, i.e.

$$Y_1 = -x_1\partial_{x_2} + x_2\partial_{x_1} = -\partial_\theta \tag{22'}$$

$$Y_2 = x_1\partial_{x_1} + x_2\partial_{x_2} + ku\partial_u = r\partial_r + ku\partial_u \tag{22''}$$

where (r, θ) are polar coordinates in the (x_1, x_2) plane; clearly, Y_1 represents a rotation in the independent variables, and Y_2 a scale transformation.

Correspondingly, we have

$$X_1 = u_\theta\partial_u \qquad X_2 = (ku - uu_r)\partial_u \tag{23}$$

and it is easy to see that

$$F(Q_1) = \{u = u(r)\}$$
$$F(Q_2) = \left\{u(x_1, x_2) = \sum_{b=0}^{k} c_b x_1^b x_2^{k-b}\right\} \tag{24}$$

(with c_b real constants) and therefore

$$F(\mathcal{G}) = \{u = \alpha r^k\} \qquad (\alpha \in \mathcal{R}). \tag{25}$$

We will consider, for the sake of clarity, only first-order polynomial PDEs, i.e. we assume

$$\Delta = \Delta(r, \theta; u, u_r, u_\theta) \tag{26}$$

this also means that

$$X_\Delta^{(1)} = \Delta\partial_u + (D_r\Delta)\partial_{u_r} + (D_\theta\Delta)\partial u_\theta \tag{27}$$

(in the following computations, we only need this first-order prolongation of X_Δ). With this, it follows from straightforward computations that

$$[X_1, X_\Delta] = -\frac{\partial\Delta}{\partial\theta} \tag{28}$$

$$[X_2, X_\Delta] = -k\Delta + r\frac{\partial\Delta}{\partial r} + ku\frac{\partial\Delta}{\partial u} + ku_\theta\frac{\partial\Delta}{\partial u_\theta} + (k-1)u_r\frac{\partial\Delta}{\partial u_r}. \tag{29}$$

Equation (28) and the polynomial assumption on Δ, guarantees that $\Delta \in \mathcal{I}_{Q_1}$ implies

$$\Delta = \gamma_{abcd} r^a u^b u_r^c u_\theta^d \tag{30}$$

moreover, (29) requires that, in order to have $\Delta \in \mathcal{I}_\mathcal{G}$, only the γ_{abcd} with

$$k(b+c+d-1) = c-a \tag{31}$$

can be nonzero.

As instances of equations satisfying these conditions we can quote

$$\left(\frac{u}{r^k}\right)^\alpha r\frac{\partial u}{\partial r} + \left(\frac{u}{r^k}\right)^\beta \frac{\partial u}{\partial\theta} = 0 \tag{30$'$}$$

$$\left(\frac{u}{r^k}\right)^\alpha \frac{1}{r^{2k-1}} u_\theta^2 u_r + \left(\frac{u}{r^k}\right)^b \frac{1}{r^{2k-3}} u_r^3 = 0 \tag{30$''$}$$

where in both cases α, β, k are arbitrary integers.

We can now check that indeed X_Δ leaves the space of functions of the form $u = \alpha r^k$ invariant. In fact,

$$\begin{aligned} X_\Delta(\alpha r^k) &= \gamma_{abcd} r^a \alpha^b r^{kb} (\alpha k)^c r^{c(k-1)} \delta_{d,0} \\ &= \beta\gamma_{abc0} r^{a+kb+c(k-1)} \end{aligned} \tag{32}$$

and it follows from (31) that the exponent of r in the final expression is just k. This shows that indeed $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$, as claimed.

Let us now consider again $\mathcal{F}$ given by the functions of unit norm, where we define

$$\|f\|^2 = \langle f, f\rangle \equiv \int_D |f(x_1, x_2)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{33}$$

(here D is the unit disk $r \leqslant 1$); in this case the only functions of $F(\mathcal{G})$ which are also in $\mathcal{F}$ are

$$f_\pm(x_1, x_2) = \pm\alpha_0 r^k \qquad \left[\alpha_0 = \sqrt{(k+1)/\pi}\right]. \tag{34}$$

We rewrite the action of X_Δ as

$$(I + \varepsilon X_\Delta)(f) = f + \varepsilon\Delta[f] \tag{35}$$

thus—for $\|f\| = 1$—the norm of this is one, i.e. X_Δ leaves $\mathcal{F}$ invariant, if and only if

$$\langle f, \Delta[f]\rangle = 0. \tag{36}$$

Note that $F(\mathcal{G})$ is a one-dimensional linear space, so that $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$ means that necessarily $X_\Delta(f) = \lambda f$, for some real number λ, for any $f \in F(\mathcal{G})$. With this, (36) means

$$\langle f, \lambda f\rangle \equiv \lambda\|f\|^2 = 0 \tag{37}$$

which can be true only for $\lambda = 0$. This shows that the condition that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ guarantees also that the isolated points in $F(\mathcal{G}) \cap \mathcal{F}$ are solutions to Δ, as claimed by our theorem.

The discussion of example 4 could be easily extended to the case where the constraint $\|f\| = 1$ is removed, essentially as in example 3 above; in this case, Λ would be the line of functions $u = \lambda r^k$.

Note that now (30′) and (30″) can be easily solved by the separation of variables, thus providing an explicit illustration of our results.

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